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$\mathcal{M}$-harmonic products*

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ABSTRACT

An explicit description is given of all pairs of holomorphic functions f and g in the open unit ball of $\mathbb{C}^n$, for which the product $f\bar{g}$ is annihilated by the Moebius-invariant Laplacian.

1. If f and g are nonconstant holomorphic functions in a connected open set $\Omega \subset \mathbb{C}$, then $f\bar{g}$ cannot be harmonic.

2. If u and u^2 are harmonic in $\Omega \subset \mathbb{C}$, then at least one of u and $\bar{u}$ is holomorphic in Ω.

These two facts are well known; their proofs are easy exercises. In the present paper we study their analogues in the open unit ball B_n of $\mathbb{C}^n$ $(n>1)$ and with "$\mathcal{M}$-harmonic" in place of harmonic.

As in [3], we say that a functdion u is $\mathcal{M}$-harmonic in B_n if

(1) $\qquad (\tilde{\Delta} u)(z)=0$

for every $z \in B_n$. The operator $\tilde{\Delta}$ is the *Moebius-invariant Laplacian*. It is uniquely characterized by the requirement that

(2) $\qquad \tilde{\Delta}(u \circ \psi)=(\tilde{\Delta} u) \circ \psi$

for every biholomorphic map ψ from B_n onto B_n, together with

(3) $\qquad (\tilde{\Delta} u)(0)=(\Delta u)(0),$

where Δ is the ordinary Laplacian.

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A calculation [3; p. 48] shows that u is $\mathcal{M}$-harmonic in B_n if and only if

$$(4)\qquad \sum_{i=1}^{n} (D_i\bar{D}_i u)(z) = \sum_{j,k=1}^{n} z_j\bar{z}_k(D_j\bar{D}_k u)(z)$$

for all $z\in B_n$. Here $D_j=\partial/\partial z_j$, $\bar{D}_k=\partial/\partial\bar{z}_k$.

It is clear from (4) that all holomorphic functions in B_n are $\mathcal{M}$-harmonic, as are the pluriharmonic ones.

The two facts stated at the start of this paper lead to two questions that involve $\tilde{\Delta}$ in place of Δ:

QUESTION 1. *If f and g are holomorphic functions in B_n, and $f\bar{g}$ is $\mathcal{M}$-harmonic, what else can one say about f and g?*

QUESTION 2. *For which $\mathcal{M}$-harmonic functions u is u^2 also $\mathcal{M}$-harmonic?*

We have only been able to find partial answers to Question 2, but our answer to Question 1 is complete and (at least as far as we were concerned) quite unexpected:

THEOREM I. *Suppose that f and g are nonconstant holomorphic functions in B_n, and that $f\bar{g}$ is $\mathcal{M}$-harmonic.*

(a) *When $n=2$, this cannot happen.*

(b) *When $n\geq 3$, then there exist*

(i) *an integer m, $2\leq m\leq n-1$,*

(ii) *a unitary transformation $U:\mathbb{C}^n\to\mathbb{C}^n$,*

(iii) *entire functions $\varphi:\mathbb{C}^{m-1}\to\mathbb{C}$ and $\psi:\mathbb{C}^{n-m}\to\mathbb{C}$, such that*

$$(5)\qquad f(Uz)=\varphi\left(\frac{z_2}{1-z_1},\dots,\frac{z_m}{1-z_1}\right),\ g(Uz)=\psi\left(\frac{z_{m+1}}{1-z_1},\dots,\frac{z_n}{1-z_1}\right).$$

Moreover, $f(B_n)=\varphi(\mathbb{C}^{m-1})$, $g(B_n)=\psi(\mathbb{C}^{n-m})$, and $(f\bar{g})(B_n)=\mathbb{C}$ or $\mathbb{C}\setminus\{0\}$. (The latter case occurs only when both φ and ψ omit the value 0.)

Conversely, a straightforward calculation shows that $\tilde{\Delta}(f\bar{g})=0$ if (5) holds.

Before we begin the proof of Theorem I we note that when $u=f\bar{g}$ (and f,g are holomorphic), then equation (4) becomes

$$(6)\qquad \sum_{i=1}^{n} (D_i f)(z)\overline{(D_i g)(z)} = \sum_{j=1}^{n} z_j(D_j f)(z) \sum_{k=1}^{n} \overline{z_k(D_k g)(z)}.$$

In terms of the *gradient*

$$(7)\qquad (\nabla f)(z)=((D_1 f)(z),\dots,(D_n f)(z))$$

and the *radial derivative* (see [3; p. 103])

$$(8)\qquad (\mathcal{R}f)(z)=\sum_{j=1}^{n} z_j(D_j f)(z),$$

(6) takes the form

$$(9) \qquad \langle \nabla f/\mathscr{R}f, \nabla g/\mathscr{R}g \rangle = 1$$

(wherever $\mathscr{R}f \neq 0$ and $\mathscr{R}g \neq 0$). Here $\langle \cdot, \cdot \rangle$ is the familiar hermitian inner product in $\mathbb{C}^n$, i.e.,

$$(10) \qquad \langle x, y \rangle = \sum_{i=1}^{n} x_i \bar{y}_i .$$

The hypothesis of Theorem I thus implies (9). This shows why the following result (stated in a more general setting than we need) is relevant to the proof of Theorem I.

THEOREM II. *Let n and p be positive integers, and suppose that F and G are holomorphic maps from a connected open set $\Omega \subset \mathbb{C}^p$ into $\mathbb{C}^n$, such that*

$$(11) \qquad \langle F(z), G(z) \rangle = 1$$

for all $z \in \Omega$.

Then there exists an integer m, $1 \leq m \leq n$, and an orthonormal basis $\{b_1, \ldots, b_n\}$ of $\mathbb{C}^n$, such that , setting $F_j = \langle F, b_j \rangle$, $G_j = \langle G, b_j \rangle$, we have
(i) *$F_j = 0$ if $m+1 \leq j \leq n$,*
(ii) *$G_j = 0$ if $2 \leq j \leq m$,*
(iii) *F_1 and G_1 are positive constants, $F_1 G_1 = 1$.*
Moreover, $m \geq 2$ and $n \geq 3$ if neither F nor G are constant.

We note another way of stating (i) and (ii), namely that F and G have the form

$$(12) \qquad \begin{cases} F = (F_1, F_2, \ldots, F_m, 0, \ldots, 0) \\ G = (G_1, 0, \ldots, 0, G_{m+1}, \ldots, G_n) \end{cases}$$

relative to the basis $\{b_1, \ldots, b_n\}$.

PROOF. Assume that $0 \in \Omega$, without loss of generality, and let β be an open ball in Ω with center at 0. The reason for introducing β is simply that $z \in \beta$ implies $\bar{z} \in \beta$. [If $z = (z_1, \ldots, z_p)$ then $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_p)$.] The main step in the proof is to show that (11) implies

$$(13) \qquad \langle F(z), G(w) \rangle = 1$$

for all $(z, w) \in \Omega \times \Omega$.

The function $\Phi(z, w) = \langle F(z), G(\bar{w}) \rangle$ is holomorphic in $\beta \times \beta$, and it satisfies

$$(14) \qquad \Phi(z, \bar{z}) = \langle F(z), G(z) \rangle = 1$$

for all $z \in \beta$. Since $(z, w) \to (z + iw, z - iw)$ is an invertible linear transformation of $\mathbb{C}^p \times \mathbb{C}^p$ which carries $R^p \times R^p$ to the set $\{(z, \bar{z}) : z \in \mathbb{C}^p\}$, (14) implies that $\Phi(z, w) = 1$ in $\beta \times \beta$. The above mentioned symmetry property of β shows therefore that $\Phi(z, \bar{w}) = 1$ in $\beta \times \beta$, i.e., that (13) holds in $\beta \times \beta$, hence also in $\Omega \times \Omega$, by real-analyticity.

Now let m be the dimension of the vector space X which is spanned by the vectors $F(z)$, where z ranges over Ω. Since $F \not\equiv 0$, $m \geq 1$, and an orthonormal basis $\{b_1, \ldots, b_n\}$ can therefore be so chosen in $\mathbb{C}^n$ that $\{b_1, \ldots, b_m\}$ spans X and the orthogonal projection of $G(0)$ into X is a multiple cb_1 of b_1, with $c > 0$ (otherwise $G(0) \in X^{\perp}$, contrary to (13)).

Conclusion (i) is now obvious.

By (13), $G(w) - G(0) \in X^{\perp}$. Thus $G_j(w) = G_j(0)$ for $1 \leq j \leq m$. It follows that $G_1(w) = c$ for all $w \in \Omega$, and that $G_j(w) = 0$ if $2 \leq j \leq m$. This gives (ii).

Next, (i) and (ii) imply

$$(15) \qquad cF_1 = F_1 \bar{G}_1 = \langle F, G \rangle = 1$$

so that F_1 is constant. Thus (iii) holds. Finally, a look at (12) shows that $m > 1$ and $n > m$ if both F and G are nonconstant.

PROOF OF THEOREM I. We have already seen that our hypothesis concerning f and g can be expressed by equation (9).

Note that if $\mathscr{R}f \equiv 0$ in B_n, then f is constant on each disc $L \cap B_n$, where L is any complex line through the center 0 of B_n, hence f is constant in B_n, contrary to our hypothesis. Thus $\mathscr{R}f \not\equiv 0$, and likewise $\mathscr{R}g \not\equiv 0$.

We can therefore apply Theorem II, with $p = n$, Ω the set of all $z \in B_n$ where $(\mathscr{R}f)(z) \neq 0$ and $(\mathscr{R}g)(z) \neq 0$, $F = \nabla f / \mathscr{R}f$, $G = \nabla g / \mathscr{R}g$, noting that then $\langle F, G \rangle = 1$ in Ω, by (9). Theorem II yields an orthonormal basis $\{b_1, \ldots, b_n\}$ of $\mathbb{C}^n$, and a constant $c > 0$, such that

$$(16) \qquad \begin{cases} \langle (\nabla f)(z), b_j \rangle = 0 & (m+1 \leq j \leq n) \\ \langle (\nabla g)(z), b_j \rangle = 0 & (2 \leq j \leq m) \\ c \langle (\nabla f)(z), b_1 \rangle = (\mathscr{R}f)(z), & \\ (1/c) \langle (\nabla g)(z), b_1 \rangle = (\mathscr{R}g)(z), & \end{cases}$$

first for all $z \in \Omega$, and then, because all functions involved in (16) are holomorphic in B_n, for all $z \in B_n$.

Now let $\{e_1, \ldots, e_n\}$ be the original (standard) orthonormal basis of $\mathbb{C}^n$, the one for which

$$(17) \qquad (z_1, \ldots, z_n) = z_1 e_1 + \cdots + z_n e_n,$$

and let $U = (u_{ij})$ be the unitary $n \times n$ matrix whose complex conjugate takes e_j to b_j, i.e.,

$$(18) \qquad b_j = \sum_{i=1}^{n} \bar{u}_{ij} e_i \quad (1 \leq j \leq n).$$

To simplify notation, we define f^*, g^* in B_n by

$$(19) \qquad f^*(z) = f(Uz), \quad g^*(z) = g(Uz).$$

If $z=(z_1,\dots,z_n)$ and $Uz=(w_1,\dots,w_n)$, then $w_i=\sum_j u_{ij}z_j$. Using this, and the chain rule, we calculate that

$$(20)\qquad (D_k f^*)(z)=\langle (\nabla f)(Uz), b_k\rangle$$

for $1\le k\le n$, and this leads to

$$(21)\qquad (\mathscr{R}f^*)(z)=(\mathscr{R}f)(Uz).$$

Analogous formulas hold of course for g and g^*, and they show that (16) turns into

$$(22)\qquad \begin{cases} (D_k f^*)(z)=0 \quad (m+1\le k\le n)\\ (D_k g^*)(z)=0 \quad (2\le k\le m)\\ c(D_1 f^*)(z)=(\mathscr{R}f^*)(z),\\ (1/c)(D_1 g^*)=(\mathscr{R}g^*)(z). \end{cases}$$

Thus f^* depends only in $z_1,\dots,z_m$, and g^* depends only on $z_1, z_{m+1},\dots,z_n$. If m were 1, then $(\mathscr{R}f^*)(z)$ would simply be $z_1(D_1f^*)(z)=c(D_1f^*)(z)$, and this forces f^* to be constant, contrary to our hypothesis. Thus $m>1$. The same argument, with g^* in place of f^*, show that $n>m$.

Thus $n\ge 3$.

The third line in (22) shows that f_1^* is constant on $L\cap B_n$, for every complex line L through the point ce_1. If $c<1$, then $ce_1\in B_n$, and therefore f^* is constant. Thus $c\ge 1$. The same argument, applied to g^*, shows that $1/c\ge 1$, hence that $c\le 1$.

We conclude that $c=1$.

Being constant on $L\cap B_n$, for every complex line L through e_1, $f^*(z)$ depends only on the ratios of the numbers $1-z_1, z_2,\dots,z_n$. Thus

$$(23)\qquad f^*(z)=\varphi\left(\frac{z_2}{1-z_1},\dots,\frac{z_m}{1-z_1}\right)$$

for some function φ, and likewise

$$(24)\qquad g^*(z)=\psi\left(\frac{z_{m+1}}{1-z_1},\dots,\frac{z_n}{1-z_1}\right)$$

for some ψ.

To finish the proof, we claim that the map $H:B_n\to\mathbb{C}^{n-1}$ given by

$$(25)\qquad H(z_1,\dots,z_n)=\left(\frac{z_2}{1-z_1},\dots,\frac{z_n}{1-z_1}\right)$$

maps H onto all of $\mathbb{C}^{n-1}$. To see this, fix $(w_2,\dots,w_n)\in\mathbb{C}^{n-1}$, put $\varepsilon=1/(1+|w|^2)$, $z_1=1-\varepsilon$, $z_j=\varepsilon w_j$ $(2\le j\le m)$. Then

$$(26)\qquad |z|^2=\sum_{j=1}^{n}|z_j|^2=(1-\varepsilon)^2+\varepsilon^2|w|^2=1-\varepsilon<1,$$

so $z \in B$, and clearly $H(z) = (w_2, \dots, w_n)$. Moreover (23) becomes now

$$(27) \qquad f^*(z_1, (1-z_1)w_2, \dots, (1-z_1)w_n) = \varphi(w_2, \dots, w_m).$$

It follows that φ must be entire, and the same is true of ψ.

This completes the proof of Theorem I.

REMARK. If f and g are as in Theorem I, then Liouville's theorem shows that neither of them can be bounded. On the other hand, it is easy to compute that in the simplest example, namely

$$(28) \qquad f(z) = \frac{z_2}{1-z_1}, \; g(z) = \frac{z_n}{1-z_1},$$

f and g belong to the Hardy space $H^2(B_n)$. A more laborious calculation (using techniques found in § 1.4 of [3]) shows that these functions are actually in $H^p(B_n)$ precisely when $p < 2n$.

DISCUSSION OF QUESTION 2. Let us now assume that u and u^2 are $\mathcal{M}$-harmonic functions in B_n.

(a) This happens of course when u is holomorphic, and also when $\bar{u}$ is holomorphic. These are the trivial cases.

(b) All nontrivial examples known to us arise from Theorem I: Let f and g be as in (5). The functions $f^r \bar{g}^s$ are also $\mathcal{M}$-harmonic, for all nonnegative integers r, s. If we multiply them by suitable constants c_{rs} and add, we reach the following conclusion:

If $E : \mathbb{C}^2 \to \mathbb{C}$ is entire and $u = E(f, \bar{g})$ then u is $\mathcal{M}$-harmonic, and so is u^2 (simply because E^2 is also entire).

(c) If u is pluriharmonic in B_n then $u = f + \bar{g}$ for some holomorphic f and g. Thus $\tilde{\Delta}(u^2) = 0$ if and only if $\tilde{\Delta}(f\bar{g}) = 0$, and Theorem I leads to the following conclusion:

If u is pluriharmonic and neither u nor $\bar{u}$ are holomorphic, then u^2 is $\mathcal{M}$-harmonic if and only if $u = f + \bar{g}$ and (5) holds.

In particular, this cannot happen when $n = 2$.

(d) If u is n times continuously differentiable on $\bar{B}_n$, then $\tilde{\Delta} u = 0$ implies that u is pluriharmonic [1], [2; p. 444], and we are back in case (c).

In particular, if $u \in C^2(\bar{B}_2)$ and $\tilde{\Delta} u = 0$, $\tilde{\Delta}(u^2) = 0$, then one of $u, \bar{u}$ is holomorphic in B_2.

We take this opportunity to point out that a more precise version of the argument used in [2; pp. 443–444] gives the following sharp result:

If $n \geq 2$, $\tilde{\Delta} u = 0$ in B_n, and the n^{th} radial derivative of u satisfies the L^2-growth condition

$$\{ \int_S |(\mathcal{R}^n u)(r\zeta)|^2 d\sigma(\zeta) \}^{1/2} = o\left(\log \frac{1}{1-r}\right)$$

as $r \nearrow 1$, then u is pluriharmonic.

Here σ is the rotation-invariant Borel probability measure on the sphere S which bounds B_n in $\mathbb{C}^n$.

(e) Note that u cannot be real-valued, since the $\mathcal{M}$-harmonic function $-[u-u(0)]^2$ would violate the maximum principle (except, of course, when u is constant).

(f) We do not know whether nontrivial examples exist in B_2 (even locally), nor do we know whether (b) and (c) exhaust all possibilities when $n \geq 3$.

(g) If u and u^2 are $\mathcal{M}$-harmonic, then so is $\Phi \circ u$, for every entire function $\Phi : \mathbb{C} \to \mathbb{C}$. This follows from the differential equation (4).

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